Proceedings of the Jangjeon Mathematical Society
25 (2022), No. 1, pp. 69 – 74
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2022.25.1.69

INITIAL VALUE PROBLEM FOR THE NEWTONIAN BOUSSINESQ APPROXIMATION WITH POWER-LAW TYPE NONLINEAR VISCOUS FLUID

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ABSTRACT. We show the existence of solutions to Newtonian Boussinesq approximation with the power-law type nonlinear viscous fluid in dimension three with $p \geq 20/9$ in the three-dimensional domains with periodic boundary conditions.

2000 AMS Subject Classification: 76A05, 76D03
Keywords : non-Newtonian fluid; Boussinesq approximation flows; existence

1. Introduction

In this paper, we study Newtonian Boussinesq approximation with the power-law type nonlinear viscous fluid:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ u_t - \nabla \cdot S + \nabla \cdot ((\rho u) \otimes u) + \nabla p = \rho \theta e_3, \\ \theta_t - \triangle \theta + (u \cdot \nabla)\theta = 0, \\ \text{div } u = 0, \end{cases} \quad \text{in } \; Q_T := \mathbb{T}^3 \times (0, T), \tag{1.1}$$

where $\mathbb{T}^3$ is a periodic domain with the spatially boundary condition, and $\rho : \mathbb{T}^3 \times (0, T) \to (0, \infty)$ is the fluid density function, $u : \mathbb{T}^3 \times (0, T) \to \mathbb{R}^3$ is the flow velocity vector, $\theta : \mathbb{T}^3 \times (0, T) \to \mathbb{R}$ is the temperature function and $p : \mathbb{T}^3 \times (0, T) \to \mathbb{R}$ is the pressure function. Also, $S = (S_{ij})_{i,j=1,2,3}$ is the stress tensor depending on the deformation tensor $Du = (\nabla u + \nabla u^T)/2$. Consider the initial value problem of (1.1), which requires initial conditions

$$\rho(x, 0) = \rho_0(x), \quad u(x, 0) = u_0(x) \quad \text{and} \quad \theta(x, 0) = \theta_0(x), \qquad x \in \mathbb{T}^3. \tag{1.2}$$

We assume that the initial data $u_0(x) \in L^2(\mathbb{T}^3)$ holds the incompressibility, i.e. div $u_0(x) = 0$.

For notational convenience, we denote by $\mathbb{M}^3_{\text{sym}}$ a vector space of all symmetric 3×3 matrices $\zeta = (\zeta_{ij})_{1 \leq i,j \leq 3}$. We note that the deviatoric stress tensor $S = (S_{ij})$, $i, j = 1, 2, 3$ satisfies the following assumptions: For $(x, t) \in Q_T$,

(i) $S : Q_T \times \mathbb{M}^3_{\text{sym}} \to \mathbb{M}^3_{\text{sym}}$ is a Carathéodory function.
(ii) (Symmetry) $S_{ij} = S_{ji}$.

Date: 2020.11.

(iii) (Polynomial growth)

$$|S_{ij}(\xi)| \le (\mu_0 + \mu_1|\xi|^{q-2})|\xi|.$$

(iv) (Coercivity condition) There exists $c_1 > 1$ such that

$$(\mu_0 + \mu_1|\xi|^{q-2})|\eta|^2 \le \frac{\partial S_{ij}}{\partial \xi_{kl}}\eta_{kl}\eta_{ij} \le c_1(\mu_0 + \mu_1|\xi|^{q-2})|\eta|^2$$

(v) (Strict monotonicity) For all $\zeta, \eta \in \mathbb{M}^3_{\rm sym}(\zeta \neq \eta)$, $S(\zeta) - S(\eta) : (\zeta - \eta) > 0$.

To motivate the conditions on the stress tensor S we recall the following examples of constitutive laws

$$S(Du) = (\mu_0 + \mu_1|Du|^{q-2})Du, \quad S(Du) = (\mu_0 + \mu_1|Du|^2)^{\frac{q-2}{2}}Du, \quad 1 < q < +\infty.$$

where $\mu_0 > 0$ and $\mu_1 > 0$ are constants (see e.g. [1], [10]).

We study a system of equations which they call a non-Newtonian Boussinesq approximation. In principle, this model is derived from replacing the linear stress-strain relation in the usual Oberbeck-Boussinesq approximation by a power-law with exponent q.

In case the density function is constant and, in particular, $S(Du) = (1+|Du|^{q-2})Du$ with $p \ge 5/2$, Malek et., al. [7] shown the existence and regularity of solutions for the initial (periodic) boundary value problem using Faedo-Galerkin techniques. Also, in case $q \ge \frac{11}{5}$, Guo and Shang in [5] proved the existence of a unique weak solution of (1.1)–(1.2). We refer to [6], [9] and [3] in case the absence of the temperature function θ.

On the other hands, when the density function is not vanished, to the best of one's knowledge, there are very few results. In this direction, the aim of this study is to prove that solutions of Newtonian Boussinesq approximation with the power-law type nonlinear viscous fluid (1.1)–(1.2) exist global-in time for the periodic space in case of the stress tensor with $q \ge 20/9$. First of all, we define a solution to the equations (1.1)–(1.2) as follows:

Definition 1.1. ***(Semi-Strong solutions)*** *Suppose that* $0 < m_0 \le \inf_{x\in\mathbb{R}^3} \rho_0(x) < \infty$, $u_0 \in (W^{1,2} \cap W^{1,q})(\mathbb{T}^3)$ *and* $h_0 \in W^{1,2}(\mathbb{T}^3)$, $q \in (1,\infty)$ *and* q' *be the Hölder conjugate of* q. *We say that* (u, h) *is a strong solution of* (1.1)–(1.2) *if* (ρ, u, b) *satisfies*

$$\rho \in L^\infty(\mathbb{T}^3 \times (0,T)), \quad \nabla u \in L^3(\mathbb{T}^3 \times (0,T)) \cap L^\infty(0,T; L^q \cap L^2(\mathbb{T}^3)),$$

$$u_t \in L^2(\mathbb{T}^3 \times (0,T)), \quad S(Du) \in L^{q'}(0,T; W^{1,q'}_{loc}(\mathbb{T}^3)), \quad \nabla\, |Du|^{\frac{q}{2}} \in L^2((0,T); L^2(\mathbb{T}^3)),$$

$$\theta_t \in L^2(\mathbb{T}^3 \times (0,T)), \quad \nabla\theta \in L^\infty(0,T; L^2(\mathbb{T}^3)), \qquad \theta \in L^2(0,T; W^{2,2}(\mathbb{T}^3)).$$

Our main result is to establish the existence of semi strong solutions for $q \ge \frac{20}{9}$.

Theorem 1.2. *Suppose that* $0 < m_0 \le \inf_{x\in\mathbb{R}^3} \rho_0(x) < \infty$, $\rho_0 \in (L^1 \cap L^\infty)(\mathbb{T}^3)$, $u_0 \in (W^{1,2} \cap W^{1,q})(\mathbb{T}^3)$ *and* $\theta_0 \in W^{1,2}(\mathbb{T}^3)$, $q \in [\frac{20}{9}, \infty)$. *Then, a semi-strong solution of* (1.1)–(1.2) *exists in the sense of Definition 1.1.*

Remark 1.3. *As proposed methodology in* [4], *we can show the existence of weak solutions to* (1.1)–(1.2) *by m-th approximated method and weak convergence technique following the arguments in* [2] *and so we omit the detailed proof (see e.g. comparing to* [11]*). We focus on the existence of semi solutions with large data.*

Remark 1.4. *If the temperature function θ is vanished, the result Theorem 1.2 is same that in* [4].

Remark 1.5. *In theorem 1.2, we don't know an existence of a strong solution exists for a degenerate case, i.e $\mu_0 = 0$, and thus we leave it as an open question. Put differently, the positive constant μ_0 is plays an important role to control the convection terms in our analysis.*

2. Proof of Theorem

In this section we introduce the notation. For $1 \le q \le \infty$, we denote by $W^{k,q}(\Omega)$ the usual Sobolev spaces, namely $W^{k,q}(\Omega) = \{f \in L^q(\Omega) : D^\alpha f \in L^q(\Omega), 0 \le |\alpha| \le k\}$. The set of $q-$th power Lebesgue integrable functions on Ω is denoted by $L^q(\Omega)$ and $L^q_{\text{loc}}(\Omega)$ indicates the set of locally $q-$th power Lebesgue integrable functions defined on Ω.

Before a proof, we recall Korn's inequality in [8].

Lemma 2.1. *Let $1 < q < \infty$. Assume that u is in $W^{1,q}(\mathbb{T}^3)$. Then*

$$\|\nabla u\|_{L^q(\mathbb{T}^3)} \le C\|Du\|_{L^q(\mathbb{T}^3)},$$

where C is a positive constant depending on q.

Proof of Theorem 1.2

First, we note ρ is bounded for all time due to the structure of the transport equation (that is, due to the maximum principle), we see that $0 < \frac{1}{C} \le \rho(\cdot, t) \le C < \infty$, $0 < t < \infty$.

Second, testing $(1.1)_3$ by θ and the divergence free condition for u, we see that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} |\theta|^2\,dx + \int_{\mathbb{T}^3} |\nabla\theta|^2\,dx = 0,$$

and thus we get

$$\|\theta(t)\|_{L^2} \le \|\theta_0\|_{L^2}, \quad 0 < t < T < \infty. \tag{2.1}$$

Again, testing $(1.1)_2$ by u, and using (2.1), Lemma 2.1 and the boundedness of ρ and θ, we find that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} |u|^2\,dx + \int_{\mathbb{T}^3} \mu_0|\nabla u|^2 + \mu_1 C|\nabla u|^q\,dx \le C\|u\|_{L^2(\mathbb{T}^3)}. \tag{2.2}$$

that is,

$$u \in L^\infty(0,T;L^2(\mathbb{T}^3)), \quad \nabla u \in L^2(0,T;L^2(\mathbb{T}^3)) \cap L^q(0,T;L^q(\mathbb{T}^3)) \tag{2.3}$$

for $0 < T < \infty$.

Third, testing $-\Delta u$ and $-u_t$ to the fluid equation of (1.1), respectively, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2} + \int_{\mathbb{T}^3} \nabla S(Du) : \nabla Du\,dx = -\int_{\mathbb{T}^3}[(\rho u\cdot\nabla)u]\cdot\Delta u\,dx + \int_{\mathbb{T}^3}\rho u e_3\cdot\Delta u\,dx, \tag{2.4}$$

and

$$\frac{1}{2}\frac{d}{dt}(\mu_0\|\nabla u\|^2_{L^2} + \mu_1\|\nabla u\|^q_{L^q}) + \int_{\mathbb{T}^3}|\theta_t|^2 dx = -\int_{\mathbb{T}^3}[(\rho u\cdot\nabla)u]\cdot u_t\,dx + \int_{\mathbb{T}^3}\rho u e_3\cdot u_t\,dx, \tag{2.5}$$

Using the property of the deformation tensor S, we note that

$$\nabla S(Du):\nabla Du=\int_0^1\frac{\partial S_{ij}(Du)}{\partial \xi_{kl}}D_{kl}(\nabla u)D_{ij}(\nabla u)\geq\Big(\mu_0+\mu_1\Big|Du\Big|^{q-2}\Big)\Big|\nabla Du\Big|^2. \tag{2.6}$$

The equations (2.4) and (2.5) with the estimation (2.6) yield

$$\frac{d}{dt}\Big(\|\nabla u(\tau)\|_{L^2}^2+(\mu_0\|\nabla u\|_{L^2}^2+\mu_1\|\nabla u\|_{L^q}^q)\Big)+\|u_t\|_{L^2}^2+\int_{\mathbb{T}^3}(\mu_0+\mu_1\Big|Du\Big|)^{q-2}\Big|\nabla Du\Big|^2$$
$$\leq C\int_{\mathbb{T}^3}|\rho|^2|u|^2|\nabla u|^2dx+C\int_{\mathbb{T}^3}|u|^2dx \tag{2.7}$$

To control the term the first term in (2.7) using the boundedness of ρ, Hölder, Young, interpolation and Sobolev embedding inequalities, and consider the following three cases: this procedure is similar to that in [4], however, we make up more calculations.

A. ($3\leq q$):

$$\begin{aligned}
I_1&\leq\|\rho\|_{L^\infty}^2\|u\|_{L^{\frac{2q}{q-2}}}^2\|\nabla u\|_{L^q}^2+\frac{1}{32}\|\nabla^2u\|_{L^2}^2\\
&\leq C\|u\|_{L^{\frac{2q}{q-2}}}^{\frac{2q}{q-2}}+\frac{1}{32}(\|\nabla^2u\|_{L^2}^2+\|\nabla u\|_{L^q}^q)\\
&\leq C\|u\|_{L^6}^{\frac{2q}{q-2}}+\frac{1}{32}(\|\nabla^2u\|_{L^2}^2+\|\nabla u\|_{L^q}^q),\quad \frac{2q}{q-2}\leq 6\\
&\leq C\|u\|_{W^{1,q}}^{\frac{2q}{q-2}}+\frac{1}{32}(\|\nabla^2u\|_{L^2}^2+\|\nabla u\|_{L^q}^q).
\end{aligned}$$

B. ($\frac{12}{5}\leq q<3$):

$$\begin{aligned}
I_1&\leq\|\rho\|_{L^\infty}^2\|u\|_{L^{\frac{2q}{q-2}}}^2\|\nabla u\|_{L^q}^2+\frac{1}{32}\|\nabla^2u\|_{L^2}^2\\
&=C\|u\|_{L^{\frac{2q}{q-2}}}^{2-\frac{4}{q}}\|u\|_{L^{\frac{2q}{q-2}}}^{\frac{4}{q}}\|\nabla u\|_{L^q}^2+\frac{1}{32}\|\nabla^2u\|_{L^2}^2\\
&\leq C\|u\|_{L^{\frac{2q}{q-2}}}^2+C\|u\|_{L^{\frac{2q}{q-2}}}^2\|\nabla u\|_{L^q}^q+\frac{1}{32}\|\nabla^2u\|_{L^2}^2\\
&=C\|u\|_{L^{\frac{2q}{q-2}}}^2(1+\|\nabla u\|_{L^q}^q)+\frac{1}{32}\|\nabla^2u\|_{L^2}^2\\
&\leq C\|u\|_{L^{\frac{3q}{3-q}}}^2(1+\|\nabla u\|_{L^q}^q)+\frac{1}{32}\|\nabla^2u\|_{L^2}^2,\quad \frac{2q}{q-2}\leq\frac{3q}{3-q}\\
&\leq C(\|u\|_{W^{1,q}}^q+1)(\|u\|_{W^{1,q}}^q+1)+\frac{1}{32}\|\nabla^2u\|_{L^2}^2.
\end{aligned}$$

C. ($\frac{20}{9}\leq q<\frac{12}{5}$):

$$\begin{aligned}
I_1&\leq C\|u\|_{L^{\frac{3q}{3-q}}}^2\|\nabla u\|_{L^{\frac{6q}{(5q-6)}}}^2+\frac{1}{32}\|\nabla^2u\|_{L^2}^2\\
&\leq C\|u\|_{L^{\frac{3q}{3-q}}}^2\|\nabla u\|_{L^q}^{\frac{5q-8}{2}}\|\nabla u\|_{L^{3q}}^{\frac{12-5q}{2}}+\frac{1}{32}\|\nabla^2u\|_{L^2}^2\\
&\leq C\|u\|_{W^{1,q}}^2\|\nabla u\|_{L^q}^{\frac{5q-8}{2}}\|\nabla u\|_{L^{3q}}^{\frac{12-5p}{2}}+\frac{1}{32}\|\nabla^2u\|_{L^2}^2
\end{aligned}$$

$$\le C\|u\|_{W^{1,q}}^{\frac{5q-4}{2}}\|\nabla u\|_{L^{3q}}^{\frac{12-5p}{2}} + \frac{1}{32}\|\nabla^2 u\|_{L^2}^2$$
$$\le C\|u\|_{W^{1,q}}^{\frac{(5q-4)q}{7q-12}} + \frac{\mu_1}{32}\mathfrak{I}_p(u) + \frac{1}{32}\|\nabla^2 u\|_{L^2}^2.$$

Here, $\mathfrak{I}_q(u) := \int_{\mathbb{R}^3}(1+|Du|)^{q-2}\frac{\partial D_{ij}}{\partial x_k}\frac{\partial D_{ij}}{\partial x_k}dx, \quad$ for $1 \le i,j,k \le 3.$

To put it shortly, through the estimates above, we obtain

$$\frac{d}{dt}\Big(\|\nabla u(\tau)\|_{L^2}^2 + (\mu_0\|\nabla u\|_{L^2}^2 + \mu_1\|\nabla u\|_{L^q}^q)\Big) + \|u_t\|_{L^2}^2 + C\int_{\mathbb{T}^3}(\mu_0+\mu_1|Du|)^{q-2}|\nabla Du|^2$$
$$\le C(\|u\|_{W^{1,2}}^{\frac{2q}{q-2}} + \|u\|_{W^{1,q}}^{\frac{(5q-4)q}{7q-12}}) + C(\|u\|_{W^{1,q}}^q + 1)(\|u\|_{W^{1,q}}^q + 1).$$

By Gronwall's Inequality with (2.2) and the relations $\frac{(5q-4)q}{7q-12} \le 2q$, $\frac{2q}{q-2} \le 2$, we obtain for ρ and u

$$\rho \in L^\infty(\mathbb{T}^3\times(0,T)), \quad \nabla u \in L^3(\mathbb{T}^3\times(0,T)) \cap L^\infty(0,T;L^q\cap L^2(\mathbb{T}^3)),$$

$$u_t \in L^2(\mathbb{T}^3\times(0,T)), \quad S(Du) \in L^{q'}(0,T;W^{1,q'}_{\text{loc}}(\mathbb{T}^3)), \quad \nabla\,|Du|^{\frac{q}{2}} \in L^2((0,T);L^2(\mathbb{T}^3)),$$

Lastly, we will improve the regularity with respect to θ. Multiplying the equations $(1.1)_3$ by $\partial_t\theta$ and $\nabla\theta$, respectively, like the previous way, we obtain

$$\frac{1}{2}\int_{\mathbb{T}^3}\Big(|\partial_t\theta|^2 + |\nabla^2\theta|^2\Big)dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\nabla\theta|^2dx \le C\int_{\mathbb{T}^3}|u|^2|\nabla\theta|^2.$$

Using Gronwall inequality with $u \in L^\infty(0,T;L^2(\mathbb{T}^3))$, $\theta_t \in L^2(0,T;L^2(\mathbb{T}^3))$ and $\theta \in L^\infty(0,T;H^1(\mathbb{T}^3)) \cap L^2(0,T;H^2(\mathbb{T}^3))$. The proof is complete.

Acknowledgments

Jae-Myoung Kim was supported by National Research Foundation of Korea Grant funded by the Korean Government (NRF-2020R1C1C1A01006521).

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